

OPTIMAL TIME-SEQUENTIAL DISTRIBUTION OF
SUPPORTING FIRE AGAINST TWO ENEMY GROUND UNITS

Myungho Oh

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THESIS

OPTIMAL TIME-SEQUENTIAL DISTRIBUTION OF
SUPPORTING FIRE AGAINST TWO ENEMY GROUND UNITS

by

Myungho Oh

September 1974

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Optimal Time-Sequential Distribution of
Supporting Fire Against Two Enemy Ground Units
by

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The optimal time-sequential distribution of supporting fire against enemy ground units in combat against attacking friendly units is studied. Lanchester-type models of warfare are combined with optimal control theory in this investigation. The optimal time-sequential fire-support policy is characterized for a specific problem. Although complete details for the determination of the optimal policy are not given, it is conjectured, on the basis of the theorems which were proved, that for this problem the optimal policy is to always concentrate all supporting fire on the same enemy unit (until supporting fire must be lifted).

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I. INTRODUCTION

The problem of fire support allocation is a problem of interest to the military tactician and planner. In this paper, we will study the problem of artillery fire support allocation against several enemy ground units. The problem is to determine an optimal time-sequential policy for distributing available fire support without wasting it. By wasting of fire support we mean overkilling, i.e., the destruction of enemy forces that does not contribute to the attainment of the friendly objective. Overkill would correspond to a state variable which represents the strength of a ground unit becoming negative. We do not want to overkill because it will be simply wasting the effort.

To determine a "good" allocation of fire support, one must consider the dynamic nature of combat in which opposing forces trade casualties over a period of time. Lanchester type models of warfare (see [15]) have been developed to provide insights into the dynamics of combat. Although combat attrition certainly is a random process, such a formulation is not mathematically tractable as far as analytic work is concerned. Therefore, we will consider deterministic Lanchester-type models in our work here. These are differential equation models of the combat attrition process. With such a dynamical system model and considering that fire support decisions are made over a period of time, one is led

to an optimal-control/differential-game formulation for the determination of optimal time-sequential fire support allocation.

Such time-sequential combat optimization problems may be cast as either optimal control problems or differential game, depending on whether or not both sides are modelled as rational decision makers. In the exploratory work at hand, we will consider the optimization of combat decision by only one of the combatants. This leads to consideration of an optimal control problem. Such a problem has been referred to as a Lanchester-type optimal control problem by Taylor (see [14]).

In a deterministic combat optimization problem, we usually encounter either a static or dynamic situation. If the situation is static, we may determine the optimal policy using the classical techniques from advanced calculus for problems that are either unconstrained or subject to equality constraints. If there are inequality constraints in the optimization problem, the methods of nonlinear programming may be used. If the combat situation is dynamic, we may apply the methods of the classic calculus of variations for problems that are either unconstrained or subject to equality constraints (see [4]). However, if inequality constraints are involved, it may be necessary to apply the recent results of optimal control theory.

The development of solutions to Lanchester-type optimal control problems is not a routine matter due to several technical difficulties. Due to physical considerations force

levels cannot become negative, and we must introduce state-variable inequality constraints (SVIC's), i.e., no state variable (force level) can be allowed to be negative. All Lanchester-type dynamic tactical allocation problems have SVIC's. The maximum principle (in its original form) is inadequate to solve a problem with SVIC's, since an extremal may contain a subarc which lies on the boundary of the state space. By an extremal we mean any trajectory on which the necessary conditions of optimality are satisfied for all t . When SVIC's are present in a problem, the necessary conditions (i.e., Pontryagin's maximum principle) need to be modified (see [7]). To simplify considerations for the problem studied in this thesis, we will assume that conditions are such that all state variables are always non-negative. Thus SVIC's are not considered further in this thesis.

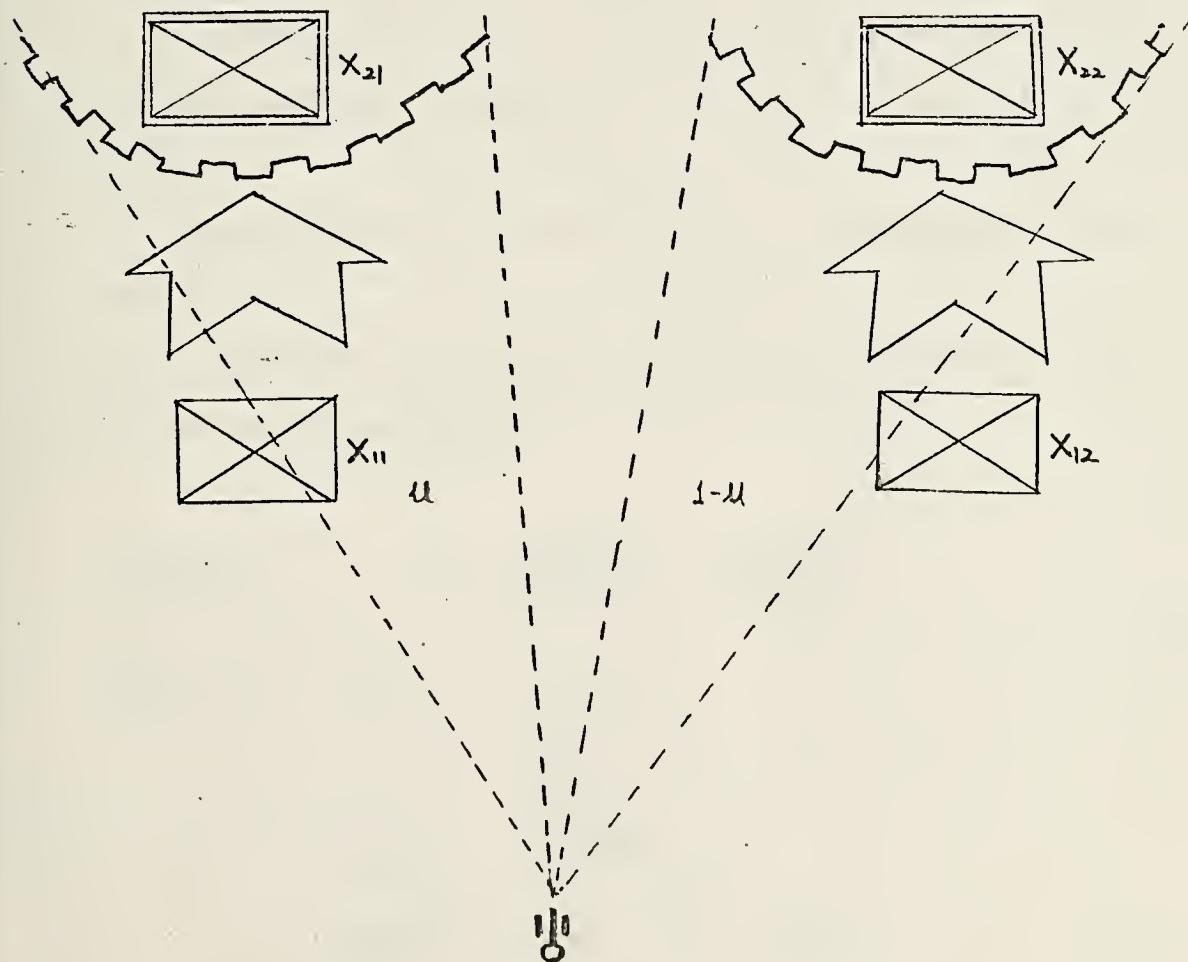
Another difficulty for this type of problem is the possible presence of a singular subarc. Consider a control problem in which, for example, a single control variable u appears linearly in the Hamiltonian H . The maximum principle fails to determine the optimal control when $\partial H / \partial u = 0$ for a finite interval of time. The corresponding trajectory is called a singular subarc. Use of the terminology singular is motivated by the fact that in such problems the matrix of second partial derivatives of the Hamiltonian with respect to the control variables is a singular matrix. We will call an extremal singular if it contains a singular subarc.

We will study the problem at hand in two phases. In the first phase we will consider nonsingular extremals, while in the second phase we will consider singular extremals. One result of the study for this problem is that there is no nice algorithm to determine the optimal control for our problem. In general, combat optimization problems dealing with multiple units are extremely complicated, especially if there are some singular subarcs.

II. ARTILLERY FIRE SUPPORT ALLOCATION MODEL

Consider combat between two ground forces Red and Blue. Each side is composed of two infantry units. Blue infantry units are being supported by artillery fire.

We want to find the best policy for allocating Blue's artillery fire support against Red to achieve the maximum effectiveness. The measure of effectiveness is taken to be the ratio of Blue infantry forces to Red at the given time. The problem is to determine an optimal policy by which the ratio of two infantry forces at the end of the battle can be maximized.



The Blue forces will be denoted as x_{11} and x_{12} (with corresponding force levels $x_{11}(t)$ and $x_{12}(t)$) and the Red forces as x_{21} and x_{22} (with corresponding force levels $x_{21}(t)$ and $x_{22}(t)$).

The following assumptions will be made for developing the model.

1. Each infantry unit is only in contact with his opposing unit.

2. The Blue infantry units are moving to contact with the Red units and due to their movement cause negligible attrition to the opposing Red forces. Furthermore, the Red defensive positions are prepared and relatively invulnerable to small arms fire during this approach phase.

3. The defensive Red units cause attrition to the opposing Blue units according to a "square law" attrition process.

4. The Blue artillery delivers area fire against the Red infantry.

With the above assumptions we may develop the following combat optimization model:

$$\text{maximize} \quad J = k_1 \frac{x_{11}(T)}{x_{21}(T)} + k_2 \frac{x_{12}(T)}{x_{22}(T)}$$

$$\text{subject to: } \dot{x}_{11}(t) = -a_1 x_{21}(t)$$

$$\dot{x}_{12}(t) = -c_1 x_{12}(t)$$

$$\dot{x}_{21}(t) = -a_2 u(t) x_{21}(t)$$

$$\dot{x}_{22}(t) = -c_2 (1 - u(t)) x_{22}(t) \quad (1a)$$

where $x_{ij}(t)$: force level at time t
 $\dot{x}_{ij}(t)$: the time derivatives of state variables $x_{ij}(t)$
 T : time at which the battle terminates
 $u(t)$: control variable
 a_i, c_i : attrition rate coefficients
 k_i : utilities assigned to ratio of survivors.

Since the criterion functional contains force ratios at the terminal time, we may formulate a problem with a state space of lower dimension by defining new state variables as

$$\frac{x_{11}}{x_{21}} = x_1 \quad \frac{x_{12}}{x_{22}} = x_2.$$

Then we can show that (1a) is equivalent to the following optimal control problem:

$$\begin{aligned}
 & \text{maximize} && J = k_1 x_1(T) + k_2 x_2(T) \\
 & \text{subject to} && \dot{x}_1 = -a_1 + c_1 u x_1 \\
 & && \dot{x}_2 = -a_2 + c_2 (1-u) x_2
 \end{aligned} \tag{1b}$$

where all constant coefficients remain same as (1a).

Let us assume that we are given the initial conditions

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0.$$

Let us assume also that we are given a specified time T_1 which is called the duration of the combat. We will assume that the battle will be terminated at time T under one of the following conditions:

Condition 1: $x_1(T) = x_2(T) = 0$, for some $T < T_1$.

Condition 2: $T = T_1$.

III. DEVELOPMENT OF NECESSARY CONDITIONS OF OPTIMALITY

To develop the necessary conditions for optimality, let us define the Hamiltonian for the problem (1b),

$$H(t, x, p, u) = p_1(-a_1 + c_1 u x_1) + p_2(-a_2 + c_2(1-u)x_2); \quad (2)$$

p_1 and p_2 are the adjoint variables, corresponding to the state variables x_1 and x_2 , respectively and, satisfying the differential equations:

$$\begin{aligned} \frac{dp_1}{dt} &= -\frac{\partial H}{\partial x_1} = -c_1 u p_1 \\ \frac{dp_2}{dt} &= -\frac{\partial H}{\partial x_2} = -c_2(1-u)p_2. \end{aligned} \quad (3)$$

By Pontryagin's maximum principle, we can show that the nonsingular optimal control will be obtained by maximizing the Hamiltonian over the control variable. Hence we may observe that the extremal control is then given by

$$u(t) = \begin{cases} 0 & \text{if } S < 0 \\ 1 & \text{if } S > 0 \\ \text{indeterminate if } S = 0, \end{cases}$$

where $S = c_1 p_1 x_1 - c_2 p_2 x_2$.

We will call u the extremal control function and S the switching function. The extremal control u so defined may furnish the global maximum, a local maximum, or only a stationary values for the criterion function.

Since the Hamiltonian H is a linear function of the control variable u , if the switching function S becomes identically zero for some finite interval of time, the maximum principle fails to provide any information about the optimal control u . In this case all the admissible values of u maximize the Hamiltonian; therefore theory of singular extremals is required to solve this problem. If $S = \frac{\partial H}{\partial u}$ is to be identically zero for some finite interval of time, then all its derivatives with respect to the time must also vanish. That is, the trajectory remains on a singular subarc if and only if

$$0 = \frac{\partial H}{\partial u} = \frac{d}{dt} \left(\frac{\partial H}{\partial u} \right) = \frac{d^2}{dt^2} \left(\frac{\partial H}{\partial u} \right) = \dots \quad (5)$$

From these conditions, we will have

$$0 = \frac{\partial H}{\partial u} = S = c_1 p_1 x_1 - c_2 p_2 x_2 \quad (6)$$

$$0 = \frac{d}{dt} \left(\frac{\partial H}{\partial u} \right) = \dot{S} = -a_1 c_1 p_1 + a_2 c_2 p_2 \quad (7)$$

$$0 = \frac{d}{dt^2} \left(\frac{\partial H}{\partial u} \right) = \ddot{S} = a_1 c_1^2 p_1 u - a_2 c_2^2 p_2 (1-u). \quad (8)$$

From (6) - (8), we have

$$u = \frac{c_2}{c_1 + c_2} = s, \quad (9)$$

defining s , the singular control.

The generalized Legendre-Clebsch condition is necessary for singular subarc to yield a maximum return (see [9]).

The condition will be given $(-1)^k \frac{\partial}{\partial u} \left[\frac{d^{2k}}{dt^{2k}} \left(\frac{\partial H}{\partial u} \right) \right] \leq 0$. For the problem at hand,

$$\frac{\partial}{\partial u} \left[\frac{d^2}{dt^2} \left(\frac{\partial H}{\partial u} \right) \right] = a_1 c_1^2 p_1 + a_2 c_2^2 p_2 \geq 0,$$

since $p_i \geq 0$. Therefore, the Legendre-Clebsch condition is satisfied.

For the singular subarc $S = \dot{S} = 0$; it follows that

$$\frac{x_1(t)}{x_2(t)} = \frac{a_1}{a_2}. \quad (10)$$

We will refer to the surfaces S_x defined by (10), and S_p , defined by (7) as singular surfaces. Singular arcs lie in these. The surface defined by

$$S = c_1 p_1 x_1 - c_2 p_2 x_2 = 0$$

is called the switching surface.

The surfaces or curves S_p and S_x can be drawn in the p and x space, respectively, but the switching surface S involves both p and x and we cannot visualize it easily.

IV. SYNTHESIS OF EXTREMAL CONTROL

A. SYNTHESIZING FUNCTION

Let X be the Euclidean space on which the state variables x and their associated adjoint variables p are defined, and U be the Euclidean space for control variables. Suppose we can find a real valued function $g: X \rightarrow U$ such that $\dot{x} = f(x, g(x, p))$ determines all the extremal trajectories from initial to terminal time. Then the optimal control function may be explicitly determined by the function of state variables and adjoint variables such that the optimal control at each time depends only on the points of the space at which the state variables and the adjoint variables are located at the given time. The function $g(x, p)$ will be called a synthesizing function and the explicit determination of such a function is called the synthesis of the extremal control, and the explicit determination of the time history of the extremal trajectory is called synthesis of extremal trajectory.

Unfortunately, an existence theorem for the synthesizing function has not been proved. Furthermore, there is no general algorithm for the problem of finding such a function. If the set of terminal values for the state variables is not explicitly specified, it may be extremely difficult to find a synthesizing function. The determination of the synthesizing function must be done on a case-by-case basis, since there is no general algorithm. In the problem at hand the terminal state is not specified. However, we know that the control

depends upon the state variable x and the adjoint variable p , and it will be determined by the maximum principle when (x, p) does not belong to the singular surface.

The difficulty is that x is only known initially and p is known only at the problem's end until we have managed to match x and p with the maximum principle to get a solution.

However we may analyze the behavior of the switching function, which is a type of synthesizing function. Although it involves both state and adjoint variables we can infer its behavior as a function of time, along an extremal, to establish several results. Further the trajectory lies on singular subarc for some finite interval of time, we may consider the singular surfaces S_x and S_p as a synthesizing function.

B. SYNTHESIS OF CONTROL FOR A TIME INTERVAL DURING WHICH THE CONTROL IS CONSTANT

For synthesizing extremal trajectory, let us consider the constant control at first. Suppose we choose a constant optimal control from the beginning to the end of the battle. Then from the differential equation (1b), we will have the following solutions.

For $u = 1$;

$$x_1(t) = \max \left[\left(x_1^0 - \frac{a_1}{c_1} \right) e^{c_1 t} + \frac{a_1}{c_1}, 0 \right]$$

$$x_2(t) = \max [x_2^0 - a_2 t, 0] \quad (12a)$$

$$p_1(t) = k_1 e^{c_1(T-t)}$$

$$p_2(t) = k_2 \quad (12b)$$

$$s(t) = c_1 k_1 x_1 e^{c_1(T-t)} - c_2 k_2 x_2 \quad (12c)$$

$$\dot{s}(t) = -a_1 c_1 k_1 e^{c_1(T-t)} + a_2 c_2 k_2 \quad (12d)$$

$$\ddot{s}(t) = a_1 c_1^2 k_1 e^{c_1(T-t)} \quad (12e)$$

For $u = 0$;

$$x_1(t) = \max \{x_1^0 - a_1 t, 0\}$$

$$x_2(t) = \max \{(x_2^0 - \frac{a_2}{c_2}) e^{c_2 t} + \frac{a_2}{c_2}, 0\} \quad (13a)$$

$$p_1(t) = k_1$$

$$p_2(t) = k_2 e^{c_2(T-t)} \quad (13b)$$

$$s(t) = c_1 k_1 x_1 - c_2 k_2 x_2 e^{c_2(T-t)} \quad (13c)$$

$$\dot{s}(t) = -a_1 c_1 k_1 + a_2 c_2 k_2 e^{c_2(T-t)} \quad (13d)$$

$$\ddot{s}(t) = -a_2 c_2^2 k_2 e^{c_2(T-t)} \quad (13e)$$

$$\text{For } u = s = \frac{c_2}{c_1 + c_2}$$

$$x_1(t) = \max \{(x_1^0 - \frac{a_1}{\alpha}) e^{\alpha t} + \frac{a_1}{\alpha}, 0\}$$

$$x_2(t) = \max \{(x_2^0 - \frac{a_2}{\alpha}) e^{\alpha t} + \frac{a_2}{\alpha}, 0\} \quad (14a)$$

$$p_1(t) = k_1 e^{\alpha(T-t)}$$

$$p_2(t) = k_2 e^{\alpha(T-t)} \quad (14b)$$

$$S(t) = (c_1 k_1 x_1 - c_2 k_2 x_2) e^{\alpha(T-t)} \quad (14c)$$

$$\dot{S}(t) = (-a_1 c_1 k_1 - a_2 c_2 k_2) e^{\alpha(T-t)} \quad (14d)$$

$$\ddot{S}(t) = \frac{c_2}{c_1 + c_2} e^{\alpha(T-t)} (a_1 c_1^2 k_1 - a_2 c_2 k_2), \quad (14e)$$

where

$$\alpha = \frac{c_1 c_2}{c_1 + c_2}.$$

We will use the above results to determine the optimal policy for more general cases.

C. DEVELOPMENT OF THEOREMS ON THE CHARACTERIZATION OF AN OPTIMAL POLICY

By analyzing the behavior of switching function, we may establish the following theorems.

THEOREM 1

Let E be a maximizing extremal which contains no singular subarc for the optimal control problem given by (1b). Then there is at most one switching in E .

Proof:

By hypothesis, since E does not contain a singular subarc, the optimal control can be determined by Pontryagin's maximum principle, except possibly at one point, as we shall see. We have got the switching function from the Hamiltonian by means of the maximum principle,

$$S(t) = c_1 p_1 x_1 - c_2 p_2 x_2.$$

Then its derivative is

$$\dot{S}(t) = a_2 c_2 p_2 - a_1 c_1 p_1$$

and

$$\ddot{S}(t) = a_1 c_1^2 p_1 u - a_2 c_2^2 p_2 (1-u).$$

It follows that

$$\ddot{S}(t) > 0 \text{ if } u = 1$$

$$\ddot{S}(t) < 0 \text{ if } u = 0.$$

Since the trajectory does not lie on singular subarc, we will have either $u = 1$ or $u = 0$ at $t = 0$. Let us consider the case $u = 1$ at $t = 0$; a similar discussion will hold if $u = 0$ initially. Since $u = 1$ at $t = 0$, $S(0) \geq 0$.

1. Let us consider first the cases in which $S(0) > 0$.

There are two sub-cases, accordingly as $\dot{S}(0) \geq 0$ or $\dot{S}(0) < 0$.

Case 1: $S(0) > 0$ and $\dot{S}(0) \geq 0$.

Since $S(0) > 0$, we must choose $u = 1$ at the beginning of the game. From (12b), $p_1(t) = k_1 e^{c_1(T-t)}$, $p_2(t) = k_2$. Therefore, $p_1(t)$ is decreasing function of t and $p_2(t)$ is constant. Hence $\dot{S}(0) \geq 0$ implies $\dot{S}(t) \geq 0$ and $S(t) > 0$, for all $t \geq 0$. Therefore $S(t)$ is a monotone increasing function, and consequently we will have $S(t) > 0$ for $t \in [0, T]$. Hence no switching occurs.

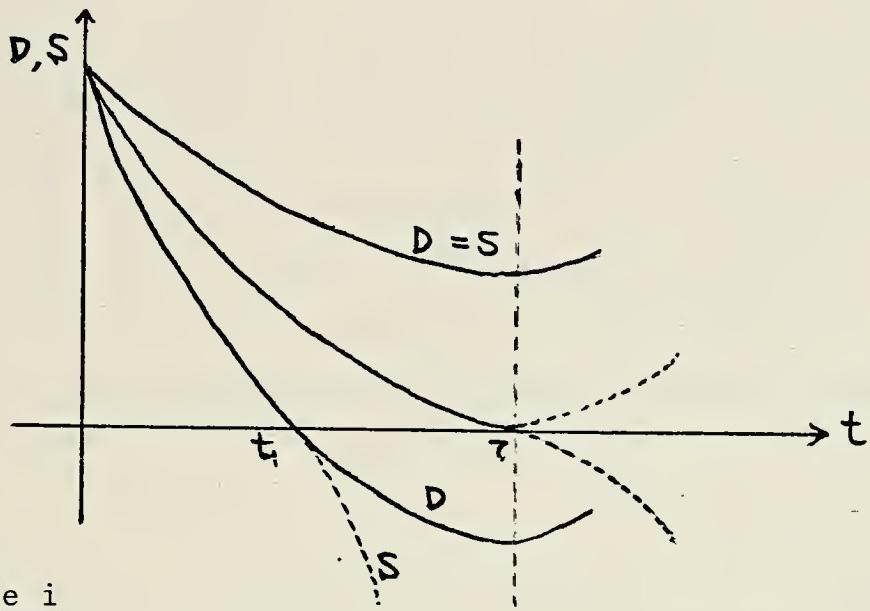
Case 2: $S(0) > 0$ and $\dot{S}(0) < 0$.

From equation (12c), S must have the form

$$D(t) = -Ae^{-c_1 t} + B \quad (15)$$

in which A and B are positive, so long as $u = 1$. But if S becomes negative u must become 0 and S will have a different form.

Let τ be the value of t for which $D = 0$. Consider also S : S may become zero at some value $t_1 < \tau$ or $S(\tau)$ may be greater than or equal to zero. There are three subcases.



Subcase i

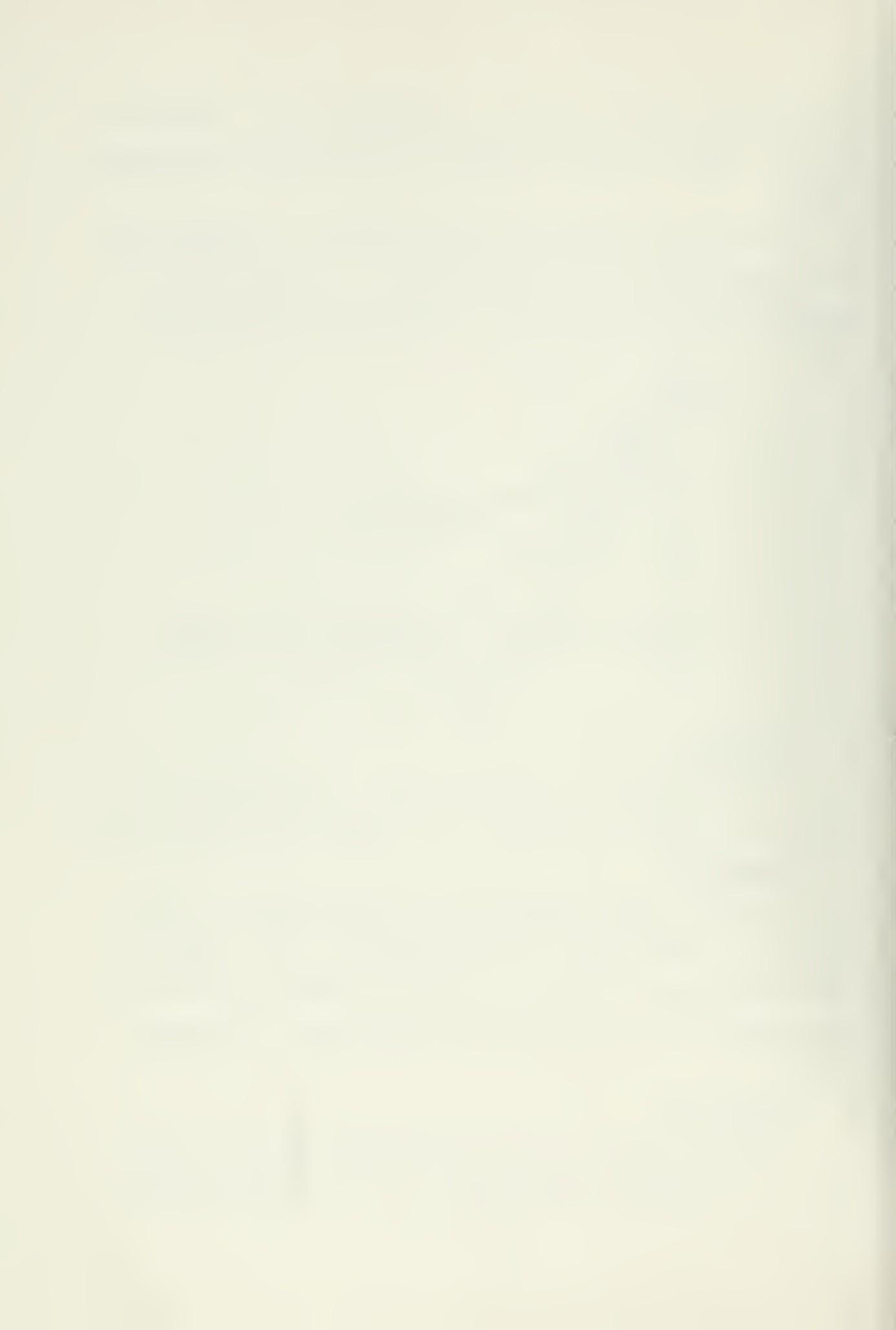
It may be that $S > 0$ for all $t \leq \tau$. In this case $S > 0$ for all t , since $\dot{S} > 0$ when $u = 1$. Hence no switching occurs.

Subcase ii

It may be that S becomes zero at some time $t_1 < \tau$ and $\dot{S}(t_1) < 0$. Then there is a switching at $t = t_1$. From the facts that $S(t_1) = 0$, $\dot{S}(t_1) < 0$ and $\ddot{S} < 0$ when $u = 0$ it follows that t_1 is a switching point, and that $S < 0$ for all $t > t_1$.

Subcase iii

It may be that t_1 in case ii coincides with τ , so that $S(t_1) = 0$ and $\dot{S}(t_1) = 0$ for some $t_1 < \tau$. Then the maximum



principle does not determine the control at time t_1 . If we choose $u = 1$ at $t = t_1$, when $\ddot{S}(t_1) > 0$. It follows that $S(t)$, $\dot{S}(t)$ are positive for $t > t_1$ and there is no switching. If we choose $u = 0$ at $t = t_1$, then by the above arguments $S(t) < 0$ for $t > t_1$ and there is one switching.

2. It may be that $S(0) = 0$. If $\dot{S}(0) > 0$ then $S(t)$ is always positive. If $\dot{S}(0) < 0$ then $S(t) < 0$ for $t > 0$ and $u \equiv 0$. Finally if $\dot{S}(0) = 0$ then we may choose $u(0)$ arbitrarily. If we choose $u = 1$ initially then $S > 0$ for $t > 0$ by argument given in case 2-iii and there is no switching. Similarly, if we choose $u(0) = 0$, S remains negative.

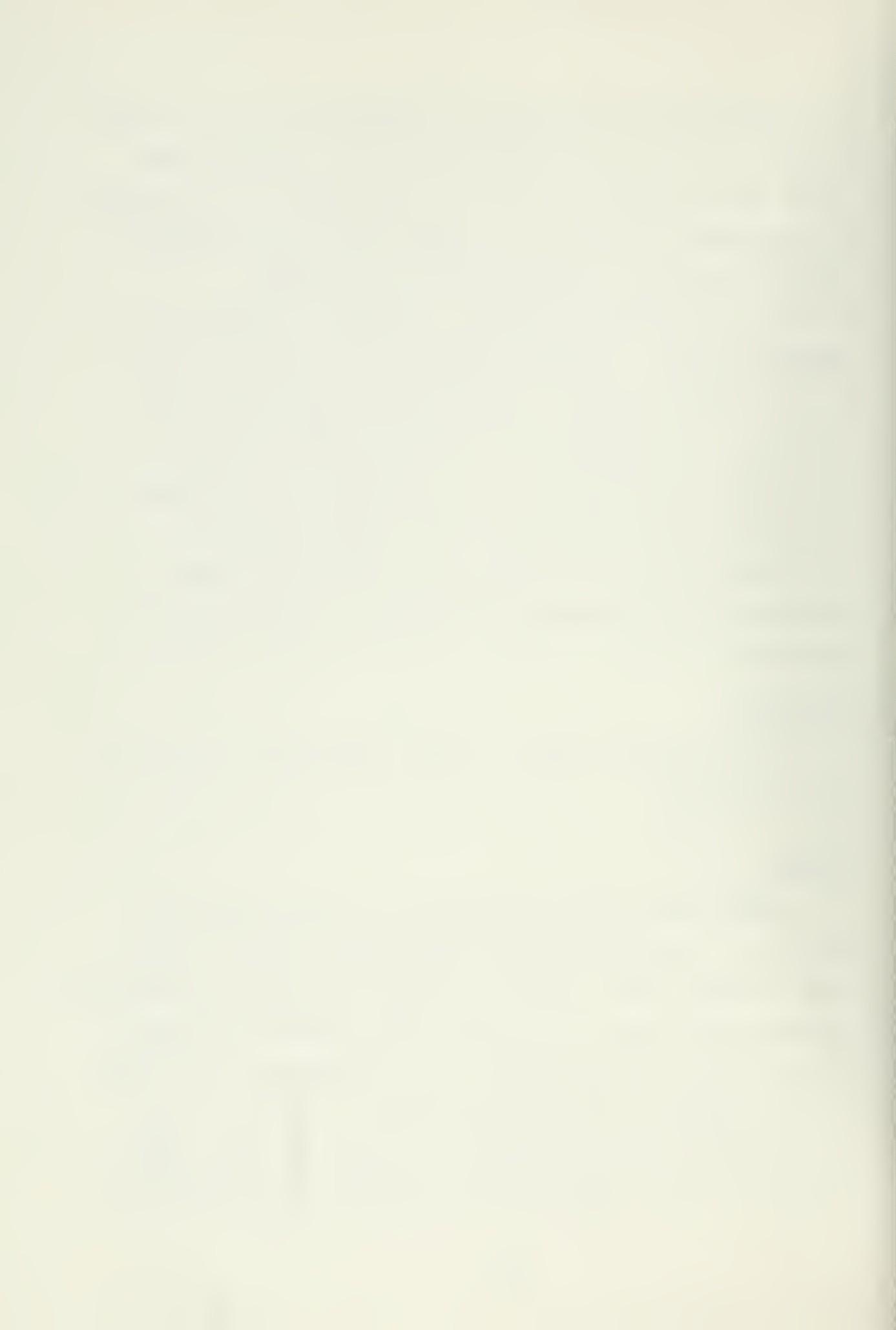
This takes care of all extremals which do not have a singular arc; a nonsingular extremal has at most one switching point.

COROLLARY:

If E contains a singular subarc, then there are at most two switchings.

Proof:

Suppose $S(0) > 0$ and there is $t_1 > 0$ such that $S(t_1) = \dot{S}(t_1) = 0$. We will show that there are at most two switchings for this case. The case for $S(0) < 0$ will be symmetrical to the case of $S(0) > 0$. Since we will choose the singular control at t_1 , we may assume that the trajectory will remain on the singular subarc until some time t_2 . If $t_2 \geq T$, we will have only one switching at t_1 from $u = 1$ to $u = \frac{c_2}{c_1 + c_2}$. Suppose $t_2 < T$. Assume that at some time t_2 we choose $u = 1$



when we are on the singular subarc. Then, by the argument given in 2-iii, S will become positive and remain positive for all $t > t_2$. Similarly if we choose $u = 0$, S will become negative and remain negative. Thus we will have at most two switchings, one at t_1 , say, when the extremal enters the singular arc and one at t_2 when it leaves it.

THEOREM 2

Consider a path E_{10} defined by the constant control $u = 1$, $0 \leq t < t_1$, and $u = 0$, $t_1 \leq t \leq T$, so that E_{10} has one switching and no singular subarc. Let J_{10} be the associated criterion function. If $x_1^0 \geq a_1/c_1$ and $x_2 \geq a_2 T$, then there is no maximizing extremal of the form E_{10} . By symmetry, there cannot be one of the form E_{01} in which $x_2^0 \geq a_2/c_2$ and $x_1^0 \geq a_1 T$. That is, a properly chosen constant control is always better than a mixed policy of this type.

Proof:

Suppose we choose the extremal control $u = 1$ for $t \in [0, t_1]$, and $u = 0$ for $t \in [t_1, T]$. Then from (12a) and (13a), we have

$$x_1(t_1) = \max \left\{ \left(x_1^0 - \frac{a_1}{c_1} \right) e^{c_1 t_1} + \frac{a_1}{c_1}, 0 \right\}$$

$$x_2(t_1) = \max \{ x_2^0 - a_2 t_1, 0 \}.$$

For $t \in [t_1, T]$,

$$x_1(t) = \max \{ x_1^0(t) - a_1(t-t_1), 0 \}$$

$$= \max \left\{ x_1^0 - \frac{a_1}{c_1} \right\} e^{c_1 t_1} + \frac{a_1}{c_1} - a_1(t-t_1), 0 \}$$

$$x_2(t) = \max\{x_2^0(t_1) - \frac{a_2}{c_2} e^{c_2(t-t_1)} + \frac{a_2}{c_2}, 0\}$$

$$= \max\{x_2^0 - a_2 t_1 - \frac{a_2}{c_2} e^{c_2(t-t_1)} + \frac{a_2}{c_2}, 0\}.$$

Suppose $x_1(t)$, $x_2(t) > 0$ for all $t \in [0, T]$; then the payoff will be

$$J_{10} = k_1[x_1^0 - \frac{a_1}{c_1} e^{c_1 t_1} + \frac{a_1}{c_1} - a_1(T - x_1)] + \quad (16)$$

$$k_2[(x_2^0 - a_2 t_1 - \frac{a_2}{c_2} e^{c_2(T-t)} + \frac{a_2}{c_2})]$$

$$J'_{10} = c_1 k_1[x_1^0 - \frac{a_1}{c_1} e^{c_1 t_1} + \frac{a_1}{c_1}] - c_2 k_2 e^{c_2(T-t_1)} (x_2^0 - a_2 t_1)$$

$$J''_{10} = c_1^2 k_1 (x_1^0 - \frac{a_1}{c_1} e^{c_1 t_1} + c_2^2 k_2 e^{c_2(T-t_1)} [x_2^0 - a_2 t_1] + \frac{a_2}{c_2})$$

Since $x_1^0 \geq \frac{a_1}{c_1}$ and $T \leq \frac{x_2^0}{a_2}$, we always have $J''_{10} > 0$. Therefore, we will have a maximum payoff at the boundary value of t_1 , i.e., $t_1 = 0$ or $t_1 = T$.

If $t_1 = 0$,

$$\max J_{10} = J_0 = k_1(x_1^0 - a_1 T) + k_2[x_2^0 - \frac{a_2}{c_2} e^{c_2 T} + \frac{a_2}{c_2}]$$

If $t_1 = T$,

$$\max J_{10} = J_1 = k_1[x_1^0 - \frac{a_1}{c_1} e^{c_1 T} + \frac{a_1}{c_1}] + k_2(x_2^0 - a_2 T).$$

In other words, we cannot improve the payoff by switching the control. We will have the same result for J_{01} , for which we switch the control from $u = 0$ to $u = 1$. We may summarize

this in the following corollary:

COROLLARY:

All nonsingular controls which maximize the criterion function for this problem have the form $u \equiv 1$ or else $u \equiv 0$.

V. DETERMINATION OF THE OPTIMAL CONTROL

A. SUFFICIENT CONDITIONS FOR OPTIMALITY

As is well known the maximum principle only provides necessary conditions of optimality. Second order sufficient conditions (see [1]) are impractical to apply to the problem at hand. Mangasarian's sufficient conditions (see [10]), unfortunately do not apply to the problem at hand.

The following is an adoption of Mangasarian's Theorem to a form more closely related to the specific problem under study. Consider the following optimal control problem.

$$\underset{u}{\text{Maximize}} \int_0^T F(t, x, u) dt + G(x_0, t_f)$$

$$\text{subject to } \dot{x} = f(t, x, u),$$

$$g(t, x, u) \geq 0,$$

$$\text{the initial conditions } C_1(x_0) \geq 0,$$

$$\text{the terminal conditions } C_2(x_f) \geq 0,$$

$$\text{where } x_0 = x(0), x_f = x(T) \text{ for specified } T.$$

THEOREM:

Assume that $F(t, x, u)$ and each component of $f(t, x, u)$ and $g(t, x, u)$ are differentiable and concave in the variable (x, u) for $t \in [0, T]$, and also each component of $C_1(x_0)$ and $C_2(x_f)$ is differentiable and concave in x_0 and x_f respectively. Then a control $\bar{u}(t)$ yields a maximum for problem (17) if there

exist $\bar{u}(t)$, $\bar{x}(t)$, $\bar{p}(t)$, $\bar{q}(t)$, $\bar{r}(t)$, $\bar{s}(t)$ such that;

$$H(t, x, u, p, q, r, s) = F(t, x, u) + p^T f(t, x, u) + q^T g(t, x, u)$$

$$+ r C_1(x_0) + s C_2(x_f)$$

$$H_x = F_x(t, \bar{x}, \bar{u}) + \bar{p}^T f_x(t, \bar{x}, \bar{u}) + \bar{q}^T g_x(t, \bar{x}, \bar{u}) + \bar{p}(t) = 0$$

$$H_u = F_u(t, \bar{x}, \bar{u}) + \bar{p}^T f_u(t, \bar{x}, \bar{u}) + \bar{q}^T g_u(t, \bar{x}, \bar{u}) = 0$$

$$G_{x_0}(\bar{x}_0, \bar{x}_f) + \bar{r} C_1(\bar{x}_0) + \bar{p}(0) = 0$$

$$G_{x_f}(\bar{x}_0, \bar{x}_f) + \bar{s} C_2(\bar{x}_f) - p(T) = 0$$

$$\bar{r} \geq 0,$$

$$\bar{r} C_1(x_0) = 0$$

$$\bar{s} \geq 0$$

$$\bar{s} C_2(x_f) = 0$$

$$\bar{q} \geq 0$$

$$\bar{q} g(t, \bar{x}, \bar{u}) = 0$$

$$\bar{p} \geq 0.$$

However, for the problem at hand (1b) it is readily seen that the right-hand sides of the differential equation are not concave functions of the state and control variables so that the above theorem cannot be applied. We will return to discuss our problem.

B. DETERMINATION OF OPTIMAL CONTROL IN TERMS OF INITIAL CONDITIONS AND TERMINAL CONDITIONS

In general we cannot determine the optimal control without some computation and comparison. However, in some cases we can establish its form easily. In this section two theorems are proved. The first one establishes a sufficient condition for some nonsingular arcs to be optimal. The second theorem establishes a necessary condition: An optimizing arc never has a terminal singular subarc.

THEOREM 3

If E does not contain a singular subarc, and if $x_1 \geq \frac{a_1}{c_1}$ and $0 < T < \min\{\frac{x_2^0}{a_2}, \frac{1}{c_1} \ln \frac{c_1 k_1 x_1^0}{c_2 k_2 x_2^0}\}$, then constant control $u = 1$ is the optimal control. By symmetry, if $x_2 \geq \frac{a_2}{c_2}$ and $0 < T < \min\{\frac{x_1^0}{a_1}, \frac{1}{c_1} \ln \frac{c_2 k_2 x_2^0}{c_1 k_1 x_1^0}\}$, then the constant control $u = 0$ is the optimal control.

Proof:

Let E be a nonsingular extremal for which $x_1 \geq \frac{a_1}{c_1}$ and $\frac{x_2^0}{a_2} > T$. Then, by theorem 2, we know that an optimal control is constant; either $u \equiv 0$ or $u \equiv 1$. Now assume $S(t) < 0$ for all $t \in [0, T]$.

From (13c), $S(0) = -c_2 k_2^0 e^{c_2 T} x_2^0 + c_1 k_1 x_1^0 < 0$. Hence $T > \frac{1}{c_2} \ln \frac{c_1 k_1 x_1^0}{c_2 k_2 x_2^0}$. Therefore, by contrapositive argument, if $T < (1/c_2) \ln(c_1 k_1 x_1^0 / c_2 k_2 x_2^0)$, then $S(0) > 0$. Thus $u = 0$ is not an optimal control, consequently the optimal control is $u = 1$. A similar argument establishes the symmetrical result.

THEOREM 4

A maximizing control does not have a singular subarc ending at $t = T$.

Proof:

Assume that we have a path which has singular control at the end, say, at least in an interval $t_1 < t < T$. Let us consider $\Delta t = T - t_1$ to be small. We want to compare this path with paths which have other controls in the interval (t_1, T) . In particular we want to compare the three trajectories which have the controls $u = s$, $u = 1$, or $u = 0$, in the final interval (t_1, T) .

Consider the Taylor expansions of J at $t = t_1$.

$$J_s = (J_s)_{t_1} + \left(\frac{dJ_s}{dt} \right)_{t_1} \cdot \Delta t + \frac{1}{2} \left(\frac{d^2 J_s}{dt^2} \right)_{t_1} \cdot \Delta t^2 + O(\Delta t^3)$$

$$J_1 = (J_s)_{t_1} + \left(\frac{dJ_1}{dt} \right)_{t_1} \cdot \Delta t + \frac{1}{2} \left(\frac{d^2 J_1}{dt^2} \right)_{t_1} \cdot \Delta t^2 + O(\Delta t^3)$$

$$J_0 = (J_s)_{t_1} + \left(\frac{dJ_0}{dt} \right)_{t_1} \cdot \Delta t + \frac{1}{2} \left(\frac{d^2 J_0}{dt^2} \right)_{t_1} \cdot \Delta t^2 + O(\Delta t^3) \quad (18)$$

Since we assume that the trajectory lies on the singular surface at the end of the battle, to have singular control on an admissible extremal at time T , it is necessary that

$$S(T) = \dot{S}(T) = 0.$$

From (6) and (7), we have

$$a_1 c_1 k_1 = a_2 c_2 k_2, \text{ i.e., } k_2 = \frac{a_1 c_1}{a_2 c_2} k_1.$$

Also

$$\frac{x_1}{a_1} = \frac{x_2}{a_2}$$

in the interval of singular control. Then at time t_1 , this condition must be satisfied.

From (10), (12a), (13a) and (14a), we may write the payoff

$$J_s = k_1 \left(\frac{a_2 + c_2}{a_2} \right) \left[\frac{a_1}{\alpha} + (x_1^1 - \frac{a_1}{\alpha}) e^{\alpha(T-t_1)} \right]$$

where $x_1^1 = x_1(t_1) > 0$, $x_2^1 = x_2(t_1) > 0$.

Similarly,

$$J_1 = k_1 \left[\frac{a_1}{c_1} + \frac{c_1}{c_2} x_1^1 - \frac{a_1 c_1}{c_2} (T-t_1) + (x_1^1 - \frac{a_1}{c_1}) e^{c_1(T-t_1)} \right]$$

$$J_0 = k_1 \left[\frac{a_1 c_1}{c_2^2} + x_1^1 - a_1 (T - t_1) + (\frac{c_1}{c_2} x_1^1 - \frac{a_1 c_1}{c_2^2}) e^{c_2(T-t_1)} \right]$$

Their derivatives with respect to t_1 are;

$$J'_s = -k_1 c_1 (x_1^1 - \frac{a_1}{\alpha}) e^{\alpha(T-t_1)}$$

$$J''_s = k_1 c_1 \alpha (x_1^1 - \frac{a_1}{\alpha}) e^{\alpha(T-t_1)}$$

$$J'_1 = -k_1 \left[-\frac{a_1 c_1}{c_2} + c_1 (x_1^1 - \frac{a_1}{c_1}) e^{c_2(T-t_1)} \right]$$

$$J''_1 = k_1 c_1^2 (x_1^1 - \frac{a_1}{c_1}) e^{c_1(T-t_1)}$$

$$J'_0 = -k_1 \left[-a_1 + (c_1 x_1^1 - \frac{a_1 c_1}{c_2}) e^{c_2(T-t_1)} \right]$$

$$J_0'' = k_1 c_1 c_2 (x_1^4 - \frac{a_1}{c_2}) e^{c_2(T-t_1)}.$$

Hence,

$$\lim_{t_1 \rightarrow T} J_s'' = k_1 c_1 (\frac{c_1 c_2}{c_1 + c_2} x_1^4 - a_1)$$

$$\lim_{t_1 \rightarrow T} J_1'' = k_1 c_1^2 (x_1^4 - \frac{a_1}{c_1})$$

$$\lim_{t_1 \rightarrow T} J_0'' = k_2 c_1 c_2 (x_1^4 - \frac{a_1}{c_2})$$

Since $(J_s')_T = (J_1')_T = (J_0')_T$, from (18) we have

$$J_1 - J_s = \frac{1}{2}(J_1'' - J_s'')(T - t_1)^2 + 0(T - t_1)^3 = \frac{k_1 c_1^3 x_1^4}{c_1 + c_2} > 0$$

$$J_0 - J_s = \frac{1}{2}(J_0'' - J_s'')(T - t_1) + 0(T - t_1)^3 = \frac{k_1 c_1 c_2^2 x_1^4}{c_1 + c_2} > 0.$$

Therefore, $J_1 > J_s$, and $J_0 > J_s$, at least if $T - t_1$ is sufficiently small.

Thus we see that if a trajectory ends with a singular subarc at the end of the battle, it cannot be an optimal trajectory.

From studying this problem we feel that following result is true. If so it would reduce this particular problem to the two cases for the optimum, i.e., $u \equiv 0$ or $u \equiv 1$. We were unable to prove this result or to find a counter example.

THEOREM 5

A maximizing arc never has a singular subarc nor a corner; that is the optimal control is always either $u \equiv 1$ or $u \equiv 0$, depending on the given conditions.

(Not proved.)

VI. DISCUSSION

Considering the theorems given above, we see that if an extremal does not contain a singular subarc, the optimal control u^* is (a) constant for all $t \in (0, T)$ and is (b) either 0 or 1. Which of these policies is better may be determined by direct computation and comparison of values of the criterion functional using the attrition-rate coefficients and the initial conditions. Thus, we see that in this case the optimal time-sequential fire-support policy for Blue is to concentrate all his artillery resources on one of the Red units for the entire duration of battle. It should be pointed out that even if an extremal contains a singular subarc, Blue's artillery fire will (optimally) still be concentrated on one of the Red units at the end of battle. This is because an optimal fire-support policy has been characterized as not terminating with a singular subarc.

We have not been able to show which policy is optimal when the trajectory contains a singular subarc for some period of time before the end of the problem. Thus, the complete explicit determination of the optimal fire support policy was not accomplished. However, the basis has been given for doing this in the future.

It should be pointed out that the insights gained into optimal fire-support policies obtained from the model (1) are no more valid than the model itself. Thus the reader should be cautioned that the optimal fire-support policy

determined here may only apply to the problem (1). As discussed by Taylor (see [14]), a combat optimization problem consists of the following three parts: (1) decision criteria, (2) model of the planning horizon (e.g. conflict termination condition), and (3) model of combat dynamics. For different objectives or different combat dynamics, the optimal fire-support policy may well be different from that given here.

VII. CONCLUSIONS

For the problem considered in this thesis, the optimal fire-support policy is piecewise constant with at most two switches. If one never divides fire between the two enemy units (singular control), then the optimal policy is constant and either 0 or 1 (i.e., concentrate all fire on one of the enemy units). Although complete details could not be worked out within the time constraints for this thesis, it is conjectured that this is the optimal policy (i.e., always concentrate all supporting fire on one of the enemy units). The reader should be cautioned that these conclusions may only apply to the specific fire support problem studied here.

BIBLIOGRAPHY

- [1.] A. E. Bryson and Y. C. Ho, "Applied Optimal Control," Blaisdell Publishing Co., Waltham, Massachusetts, 1969.
- [2.] G. M. Ewing, "Calculus of Variations with Applications," W. E. Norton and Co. Inc., New York, 1969.
- [3.] F. D. Faulkner, "Newton's Method Applied to the Solution of Classical Problems in Calculus of Variation and Differential Games," University of Michigan, 1969.
- [4.] F. D. Faulkner, "Optimum Ship Routing," Naval Postgraduate School, 1962.
- [5.] R. Issacs, "Differential Games," John Wiley and Sons, Inc., New York, 1965.
- [6.] D. H. Jacobson and J. L. Speyer, "Necessary and Sufficient Conditions for Optimality for Singular Control Problem," J. Math. Anal. Appl. 34, 239-266, 1971.
- [7.] D. H. Jacobson and M. M. Lele, "Necessary Condition of Optimality for Control Problems with State-Variable Inequality Constraints," J. Math. Anal. Appl. 35, 255-284, 1971.
- [8.] C. D. Johnson and J. E. Gibson, "Singular Solution in Problems of Optimal Control," IEEE, Transactions on Automatic Control, 4-15, 1963.
- [9.] H. J. Kelley, R. E. Kopp, and H. G. Moyer, "Singular Extremals," in Topics in Optimization, G. Leitman (ed.), Academic Press, New York, 1967, p. 63-101.
- [10.] O. L. Mangasarian, "Sufficient Conditions for the Optimal Control of Nonlinear Systems," J. SIAM Control Vol. 4, No. 1, 1966.
- [11.] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, "The Mathematical Theory of Optimal Processes," John Wiley and Sons, Inc., New York, 1962.
- [12.] G. E. Pugh and J. P. Mayberry, "Analysis of General-Purpose Forces Procurement, Lambda Corporation, Paper 59, August 1971.

- [13.] J. G. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Research Logistics Quarterly, Vol. 20, No. 4, 673-697, 1973.
- [14.] J. G. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Processes," to appear in Proceedings of The Conference on the State-of-the-Art of Mathematics in Combat Model.
- [15.] H. K. Weiss, "Lanchester-Type Models of Warfare," Proceedings of the First International Conference Operations Research, Oxford, September 1957.

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